THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 2

1

$$
\begin{aligned}
|\sum_{i=1}^{n} a_i b_i|^2 =& (\sum_{i=1}^{n} a_i b_i)(\sum_{i=1}^{n} \overline{a_i b_i}) \\
=& \sum_{i=1}^{n} |a_i b_i|^2 + \sum_{i,j=1, i\neq j}^{n} a_i b_i \overline{a_j b_j} \\
=& \sum_{i=1}^{n} |a_i b_i|^2 + \sum_{1\leq i<j\leq n} a_i b_i \overline{a_j b_j} + \sum_{1\leq i<j\leq n} a_j b_j \overline{a_i b_i} \\
=& \sum_{i=1}^{n} |a_i b_i|^2 + \sum_{i\neq j} |a_i b_j|^2 - \sum_{i\neq j} |a_i b_j|^2 + \sum_{1\leq i<j\leq n} a_i b_i \overline{a_j b_j} + \sum_{1\leq i<j\leq n} a_j b_j \overline{a_i b_i} \\
=& (\sum_{i=1}^{n} |a_i|^2)(\sum_{i=1}^{n} |b_i|^2) - (\sum_{i\neq j} |a_i b_j|^2 - \sum_{1\leq i<j\leq n} a_i b_i \overline{a_j b_j} - \sum_{1\leq i<j\leq n} a_j b_j \overline{a_i b_i}) \qquad (1)
\end{aligned}
$$

On the other hand, note that

$$
\begin{aligned}
\sum_{1\leq i<j\leq n} |a_i \overline{b_j} - a_j \overline{b_i}|^2 =& \sum_{1\leq i<j\leq n} (a_i \overline{b_j} - a_j \overline{b_i})(\overline{a_i} b_j - \overline{a_j} b_i) \\
=& \sum_{1\leq i<j\leq n} (|a_i b_j|^2 + |a_j b_i|^2 - a_i \overline{a_j} b_i \overline{b_j} - a_j \overline{a_i} b_j \overline{b_i}) \\
=& \sum_{i\neq j} |a_i b_j|^2 - \sum_{1\leq i<j\leq n} a_i b_i \overline{a_j b_j} - \sum_{1\leq i<j\leq n} a_j b_j \overline{a_i b_i} \qquad (2)
\end{aligned}
$$

Combining (1) and (2), we have the Lagrange's identity:

$$
|\sum_{i=1}^{n} a_i b_i|^2 = (\sum_{i=1}^{n} |a_i|^2)(\sum_{i=1}^{n} |b_i|^2) - \sum_{1\leq i<j\leq n} |a_i \overline{b_j} - a_j \overline{b_i}|^2
$$

2

$$
\begin{aligned}
\int_{|z|=r} y dz =& \int_0^{2\pi} r \sin\theta (ire^{i\theta}) d\theta \\
=& r^2 i \int_0^{2\pi} (\sin\theta\cos\theta + i\sin^2\theta) d\theta \\
=& r^2 i \int_0^{2\pi} \frac{\sin 2\theta}{2} d\theta - r^2 \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta \\
=& -r^2\pi
\end{aligned}
$$

3

$$\begin{aligned}\int_{|z|=1} z^m \overline{z^n} dz =& \int_0^{2\pi} e^{im\theta} e^{-in\theta}(ie^{i\theta})d\theta \\ =& i\int_0^{2\pi} e^{i(m-n+1)\theta} d\theta \\ =& \begin{cases} i\int_0^{2\pi} d\theta & \text{when } (m-n+1)=0 \\ i\left[\dfrac{e^{i(m-n+1)\theta}}{i(m-n+1)}\right]_0^{2\pi} & \text{when } (m-n+1)\neq 0 \end{cases} \\ =& \begin{cases} 2\pi i & \text{when } (m-n+1)=0 \\ 0 & \text{when } (m-n+1)\neq 0 \end{cases}\end{aligned}$$

4 By triangle inequality, for $|z| = R$, we have

$$|3z-1| \leq |3z|+1 = 3R+1 \text{ and}$$
$$|z^4+4z^2+3| = |z^2+1||z^2+3| \geq (|z|^2-1)(|z|^2-3) = (R^2-1)(R^2-3)$$

The above inequalities imply

$$\left|\int_{|z|=r} \frac{3z-1}{z^4+4z^2+3}\right| \leq \text{length of the contour} \times \frac{3R+1}{(R^2-1)(R^2-3)} = \frac{2\pi R(3R+1)}{(R^2-1)(R^2-3)}$$

5 Along the vertical line segment from R to $R+4\pi i$ with $R>0$, we have $|e^z| = e^R$. Furthermore, by triangle inequality,

$$|1+e^{3z}| \geq |e^{3z}|-1 = e^{3R}-1$$

As a result,

$$\left|\int_{|z|=\gamma_R} \frac{2e^z}{1+e^{3z}}\right| \leq \text{length of the contour} \times \frac{2e^R}{e^{3R}-1} = \frac{8\pi e^R}{e^{3R}-1}$$

6 (a) No. It is because if antiderivative exists, the contour integral of $f(z)$ along any closed contour must be zero. However, by direct calculation,

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{1}{z}dz = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = \int_0^{2\pi} i d\theta = 2\pi i =\neq 0$$

(b) Yes. The antiderivative of $g(z) = \dfrac{1}{z^2}$ is given by $G(z) = \dfrac{-1}{z}$ on $\mathbb{C}\backslash 0$.

7 Write $f(z) = f(x,y) = u(x,y)+iv(x,y)$. Since $f(z)$ is an analytic function on its domain, u and v are real differentiable and the Cauchy Riemann equations are satisfied.

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x \tag{3}$$

By definition, we have $g(z) = \overline{g(\bar{z})} = u(x,-y) - iv(x,-y)$. Write $g(z) = p(x,y)+iq(x,y)$. From this one can check that

$$p_x = u_x(x,-y), p_y = -u_y(x,-y), q_x = -v_x(x,-y), q_y = v_y(x,-y)$$

By (3), we have $p_x = q_y$ and $p_y = q_x$. Since p and q are real differentiable and the Cauchy Riemann equations are satisfied, g is analytic.